A perturbed gamma process with random effect and state-dependent error

B. Castanier[1], N. Esposito[1], M. Giorgio[2] and A. Mele[3]

[1]Université d'Angers/Laris, Angers, France, [2]Università di Napoli Federico II, Napoli, Italia, [3]Kineton, Napoli, Italia

Motivation of the study

In the literature dealing with stochastic modelling of degradation, two main forms of variability are usually distinguished: the "temporal variability" and the "unit to unit variability".

- The first form of uncertainty is associated with the progression of the degradation over time
- The second one describes the differences (i.e., heterogeneity) in the evolution of the degradation paths of different units caused by factors whose values vary from unit to unit.

In this work we consider a third form of uncertainty that is associated with the presence of measurement errors, an experimental situation that is often encountered in the practice, especially when data are collected through in-service and/or non-destructive inspections. The temporal variability is modeled by using a gamma process. The presence of unit to unit variability is accounted for by assuming that the scale parameter of the gamma process varies randomly from unit to unit. The presence of measurement error is tackled by using the hidden Markov theory.

Computations are developed using a particle filter method.

The perturbed gamma process with random effect

The perturbed model has been formulated as $Z(t) = W(t) + \varepsilon(t)$, where $W(t)$ is the actual (hidden) degradation level, $\varepsilon(t)$ is the measurement error, and $Z(t)$ is the observed (perturbed) degradation level.

Following the standard hidden Markov process theory, it is assumed that for any $\nu > 1$, any $j = 1, \ldots, n$ and any set of times $t_1, \ldots, t_n$ the measurement error $\varepsilon(t_j)$ given $W(t_j) = w_j$ is conditionally independent both on $\varepsilon(t_1), \ldots, \varepsilon(t_{j-1}), \varepsilon(t_{j+1}), \ldots, \varepsilon(t_n)$ and $W(t_1), \ldots, W(t_{j-1}), W(t_{j+1}), \ldots, W(t_n)$

The hidden degradation process $\{W(t); t \geq 0\}$ is assumed to be a non-homogeneous gamma process with random effect [Lawless and Crowder (2004), Giorgio et al. (2019)].

The pdf of the increment $\Delta W(t_1, t_2) = W(t_2) - W(t_1)$ is:

$$f_{\Delta W(t_1,t_2)}(\Delta w_{1,2}|\lambda) = \frac{\lambda^{\eta(t_1,t_2)-1} \Delta w_{1,2}^{\eta(t_1,t_2)-1}}{\Gamma\eta(t_1,t_2)]} e^{-\lambda \Delta w_{1,2}} \qquad g(\lambda) = \frac{a^b \lambda^{b-1}}{\Gamma(b)} e^{-a\lambda}, \qquad \lambda > 0$$

Where $\Gamma(\cdot)$ denotes the complete gamma function, $\eta(t)$ is the age function, which in this paper is assumed to be $\eta(t) = \left(\frac{t}{c}\right)^d$, and $\eta(t_1, t_2) = \eta(t_2) - \eta(t_1)$.

As in [Giorgio et al. (2019)], it is assumed that the measurement error $\varepsilon(t)$ depends in stochastic sense on the actual degradation level $W(t)$ and that, given $W(t) = w$ its conditional pdf is:

$$f_{\varepsilon(t)|W(t)}(\varepsilon|w) = \frac{[\alpha(w)]^{\beta(w)} (\varepsilon + w)^{-\beta(w)-1}}{\Gamma[\beta(w)]} e^{-\frac{\alpha(w)}{\varepsilon + w}}, \qquad \varepsilon \geq -w$$

Model Calibration

Model parameters have been estimated from degradation data collected via periodic inspections, considering that a random effect exists.

Estimates have been obtained by maximizing the following likelihood function $L(\boldsymbol{\theta}, \boldsymbol{z})$:

$$L(\boldsymbol{\theta}; \boldsymbol{z}) = \prod_{i=1}^{m} \prod_{j=1}^{n_i} f_{Z_{i,j}|\mathbf{Z}_{i,j-1}}(z_{i,j}|\mathbf{z}_{i,j-1})$$

where $t_{i,j}$ $(i = 1, \ldots, m; j = 1, \ldots, n; n_i \geq 1)$ indicates the time at which the $j - th$ inspection on the unit i is performed, $\boldsymbol{Z}_{i,j} = \{Z_{i,1}, \ldots, Z_{i,j}\}$ is the set of perturbed measurements made on the unit i up to the time $t_{i,j}$, $\boldsymbol{z}_{i,j} = \{z_{i,1}, \ldots, z_{i,j}\}$ denotes its realizations, $\boldsymbol{\theta}$ denotes the vector of model parameters, $t_{i,0} = 0$, and $\boldsymbol{Z}_{i,0}$ is an empty set.

Under the proposed perturbed model, the pdf in the likelihood function can be computed, for any $i = 1, \ldots, m$ and $j = 1, \ldots, n_i$ via the following recursive equations:

$$f_{W_{i,j}|\mathbf{Z}_{i,j-1}}(w_{i,j}|\mathbf{z}_{i,j-1}) = \int_0^{w_{i,j}} f_{\Delta W_{i,j}}(w_{i,j} - w_{i,j-1}) \cdot f_{W_{i,j-1}|\mathbf{Z}_{i,j-1}}(w_{i,j-1}|\mathbf{z}_{i,j-1}) dw_{i,j-1}$$

$$f_{Z_{i,j}|\mathbf{Z}_{i,j-1}}(z_{i,j}|\mathbf{z}_{i,j-1}) = \int_0^{\infty} f_{Z_{i,j}|W_{i,j}}(z_{i,j}|w_{i,j}) f_{W(i,j)|\mathbf{Z}_{i,j-1}}(w_{i,j}|\mathbf{z}_{i,j-1}) dw_{i,j}$$

$$f_{W_{i,j}|\mathbf{Z}_{i,j}}(w_{i,j}|\mathbf{z}_{i,j}) = \frac{f_{Z_{i,j}|W_{i,j}}(z_{i,j}|w_{i,j}) f_{W_{i,j}|\mathbf{Z}_{i,j-1}}(w_{i,j}|\mathbf{z}_{i,j-1})}{f_{Z_{i,j}|\mathbf{Z}_{i,j-1}}(z_{i,k}|\mathbf{z}_{i,j-1})}$$

Where $W(i, j)$ denotes the actual degradation level of the unit i at the time $t_{i,j}$, $\Delta W_{i,j} = W_{i,j} - W_{i,j-1}$ denotes the degradation increment of the unit i in the time interval $(t_{i,j-1}, t_{i,j})$, $w_{i,j}$ denotes the (unobservable) realization of $W_{i,j}$ and $\Delta w_{i,j} = w_{i,j} - w_{i,j-1}$ the unobservable realization of $\Delta W_{i,j}$.

Remaining useful life

It is assumed that a unit (conventionally) fails when its degradation level exceeds a threshold limit w_M and that failure is not self-announcing. Under these assumptions, the remaining useful life $RUL(t)$ is defined as the non-negative random variable which measures the remaining time from t up to the degradation process $\{W(t); t \geq 0\}$ first passes w_M. In the case $\{W(t); t \geq 0\}$ passes w_M at or before t, $RUL(t) = 0$.

Under this assumption, the complementary Cdf of $RUL(t)$ is defined as the conditional probability that the unit fails within $t + \tau$, given $\boldsymbol{Z}_t = \boldsymbol{z}_t$.

Hence, since $\{W(t); t \geq 0\}$ is a monotonically increasing process, the complementary Cdf of $RUL(t)$ is given by:

$$F_{RUL(t)}(\tau|\mathbf{z}_t) = 1 - P(RUL(t) \leq \tau|\mathbf{z}_t) = \int_0^{w_M} F_{W_{t+\tau}|W_t}(w_M|w_t) f_{W_t|\mathbf{Z}_t}(w_t|\mathbf{z}_t) dw_t = F_{W_{t+\tau}|\mathbf{Z}_t}(w_m|\mathbf{z}_t)$$

Where $W_{t+\tau}$ is the actual degradation level $t + \tau$ and W_t is the actual degradation level at t.The complementary Cdf of the $RUL(t)$ is here alternatively referred to as (perturbed measurement based) residual reliability.

Computational details

Unfortunately, due to the presence of measurement error, neither the likelihood function nor the $RUL(t)$ can be expressed in closed form. Thus, a numerical method must be used. In this paper, we used a particle filter method, which allows computing the required functions by using a sequential Monte Carlo based approach, once the vector of model parameters $\boldsymbol{\theta}$ is set to a plausible value.

Application example

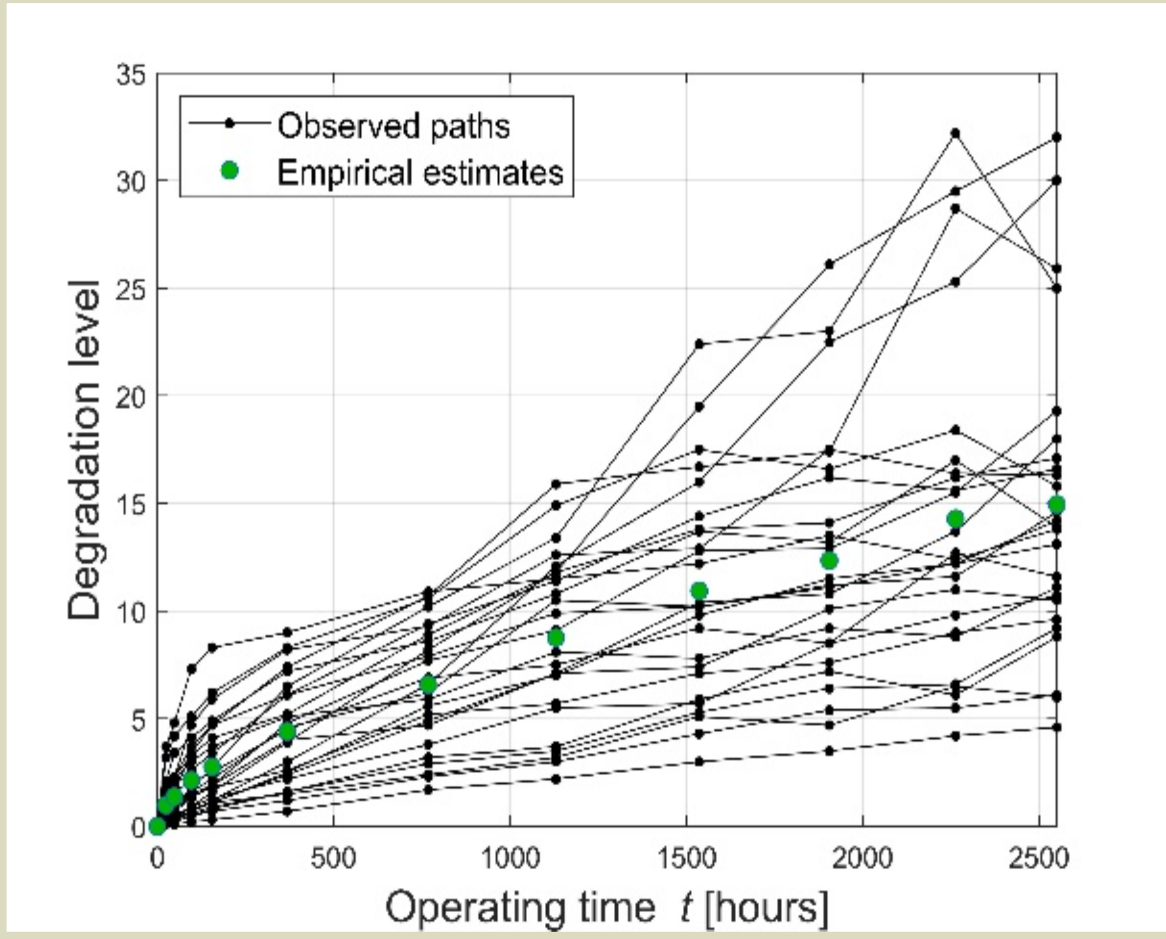

Figure 1 – Observed degradation paths of GaAs/GaAs IRLEDs and empirical estimates of $E[Z(t)]$ at the measurement times.

Data consist of the observed degradation (in terms of variation ratio of luminous power) of $m = 25$ GsAs/GaAs IRLEDs operating at 170 mA, each with $n_i = 11$ degradation measurement, performed at the same times for all the units. Here, an IRLED is assumed to fail when its degradation level passes the threshold value $w_M = 0.35$.

The considered IRLEDs data present two important features: (a) the presence of negative increments, (b) empirical evidence of a positive correlation between increments (e. g., consider 8 out of 10 empirical estimates of the correlation coefficient between consecutive increments are positive). The first feature inhibits the use of a gamma process and/or other degradation processes with intrinsically non-negative increments, the second one suggests the (possible) presence of random effect.

		Model #	
		M3	M4
MLE	$\hat{\lambda}$	/	1.37
	$\hat{a}$	2.39	/
	$\hat{b}$	6.85	/
	$\hat{c}$	7.37	14.55
	$\hat{d}$	0.62	0.58
	$\hat{\gamma}$	147.9	156.6
AIC		710.6	775.2
$lnL(\hat{\boldsymbol{\theta}}, \boldsymbol{z})$		−350.3	−383.6

In this table, the proposed model (indicated as M3) is compared with a model that neglects the presence of the random effect (indicated as M4).

The table contains the maximum likelihood estimates of models parameters, the values of Akaike information criterion indexes (AIC) and the values of the log-likelihood function computed at the MLEs.

The AIC values give evidence that according to the Akaike information Criterion the model M3 should be preferred to the model M4.

Results

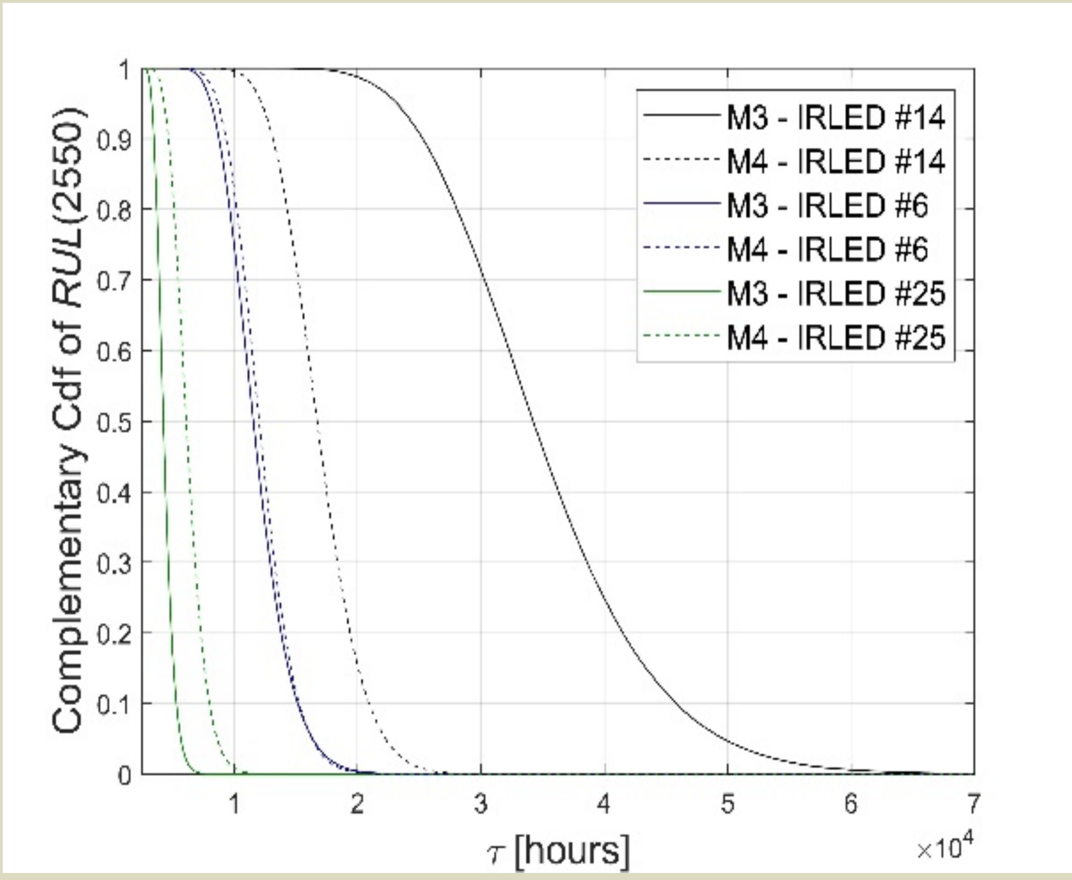

Figure 2 - MLEs of the complementary Cdf of $RUL(2550)$ of IRLEDs units #6, #14, and #25 under the models M3 and M4.

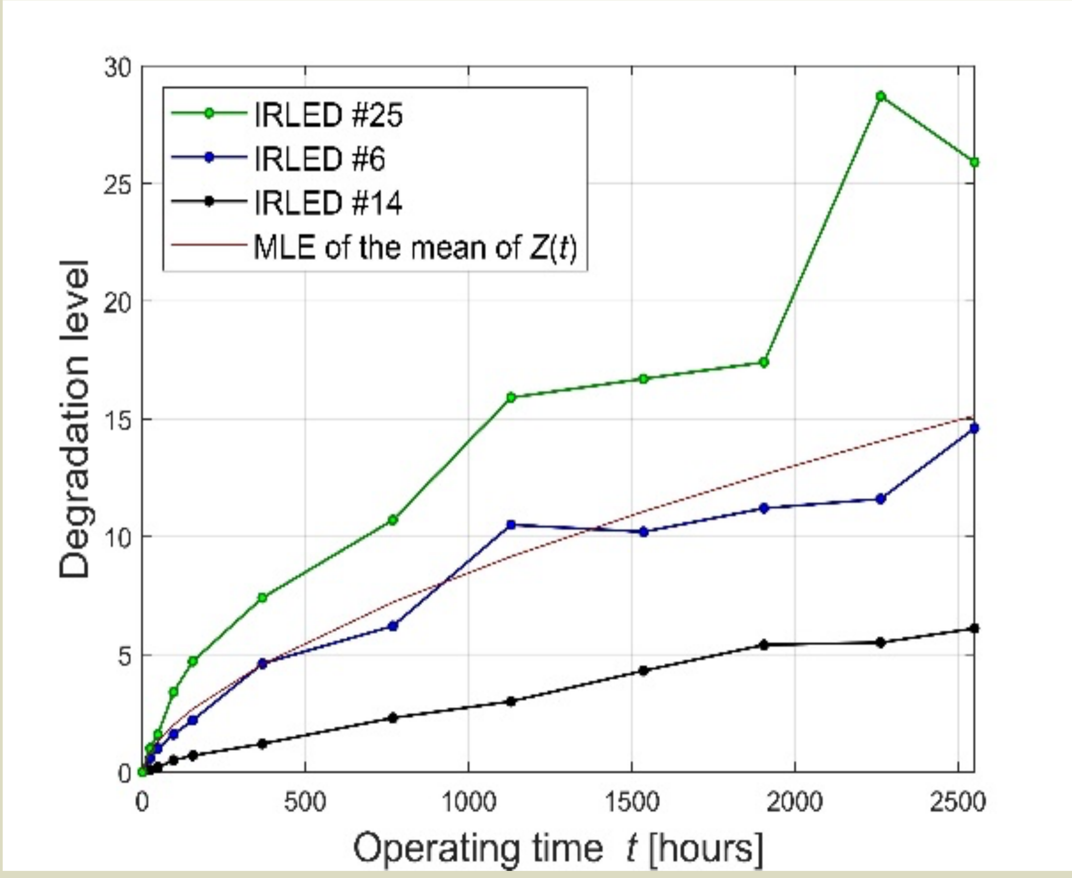

Figure 3 - Degradation measurements of IRLEDs #6, #14, and #25 and MLE of $E[Z(t)]$ obtained under the model M3.

The main purpose of the application is to analyze the repercussion of neglecting the presence of the random effect on the estimates of the cumulative distribution of the remaining useful life of the considered degrading units.

Figure 2 displays the MLEs of the complementary Cdf of $RUL(t)$ (i.e., the measurement-based residual reliability) of IRLEDs #6, #14, and #25, at the last inspection time t=2550. Solid lines are MLEs obtained under the model M3, dotted lines are those obtained under the model M4. The figure shows that neglecting the presence of the random effect leads to underestimate the complementary Cdf of $RUL(2550)$ in the case of the IRLED #14, whose degradation measurements (see Figure 3) are all below the MLE of $E[Z(t)]=E[W(t)]$ and to overestimate it in the case of the IRLED #25, whose degradation measurements are all above the MLE of $E[Z(t)]$. The figure also shows that, in the case of the IRLED #6, whose degradation measurements are close to the MLE of $E[Z(t)]$, the estimates of complementary Cdf of $RUL(2550)$ obtained under the two considered models are very similar.

References

- Giorgio M., A. Mele, and G. Pulcini (2019). A perturbed gamma degradation process with degradation dependent non-Gaussian measurement errors, *Applied Stochastic Models in Business and Industry 35(2),* 198-210.
- Lawless, J., and M. Crowder. (2004). Covariates and random effects in a gamma process model with application to degradation and failure. *Lifetime Data Analysis 10 (3)*, 213–227.